Spheres of Planes in Generalized Quaternions

Ian Gallagher, Andy Haase, Bailey Wickham, Dr. Eric Brussel

Department of Mathematics, California Polytechnic State University, San Luis Obispo

Abstract

A great discovery of Hamilton was that $\mathbb{R}^4$ had the structure of a number system, now known as the quaternions denoted by $\mathbb{H}$. Some of the planes in $\mathbb{R}^4$ are complex planes under the induced multiplication. We found that the set of such planes is naturally represented by the points on a sphere, so that sphere is what's called a *moduli space*.

Everyone who has taken Linear Algebra is familiar with the set of 2x2 matrices $M_2(\mathbb{R})$, with its two operations, addition and matrix multiplication. This is what's called a twisted form of $\mathbb{H}$, because it too is a 4-dimensional $\mathbb{R}$-vector space, which becomes isomorphic to $\mathbb{H}$ over $\mathbb{C}$. This suggests it too should be labeled a number system.

Like $\mathbb{H}$, $M_2(\mathbb{R})$ contains a set of planes which are themselves number systems under matrix multiplication, though here there are three different types. We constructed a moduli space for these planes as well. This moduli space naturally gives a probability distribution for the planes based on their type.

$$\mathbb{H} = \left\{ t + xi + yj + zk : \begin{array}{r} t, x, y, z \in \mathbb{R}, \\ i^2 = j^2 = -1, \\ ij = -ji = k \end{array} \right\}$$

Introduction

We are all familiar with $\mathbb{R}^1$ (the line), $\mathbb{R}^2$ (the plane), $\mathbb{R}^3$ (3-space), and $\mathbb{R}^4$ (4-space). Notice that each space contains many lower dimensional subspaces. For example $\mathbb{R}^4$ contains many 2D subspaces, i.e. planes.

Our study of the quaternions was initially motivated by a question: The quaternions contain many copies of the complex plane, is there a way to "see" all copies at once? Such a visual representation is called a *moduli space*.

As an example of a moduli space, to "see" the oriented lines through the origin in the plane, we can identify each line with its point of intersection with the standard unit circle. In this way the circle gives us a way to "see" all these oriented lines. We want to repeat this same idea for the planes in 4D space.

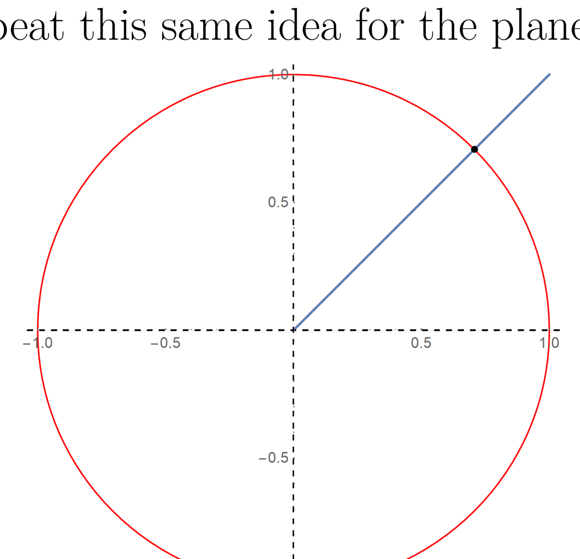

Figure: Moduli Space of Lines in Plane

Commutative Planes in the Quaternions

We started with an attempt to visualize all of the complex planes inside of $\mathbb{H}$ by representing them as points in a moduli space. We found that the oriented complex planes in $\mathbb{H}$ are naturally represented by points on a sphere, where each point represents the "i" of a different complex plane.

Moduli Space of Planes in Quaternions

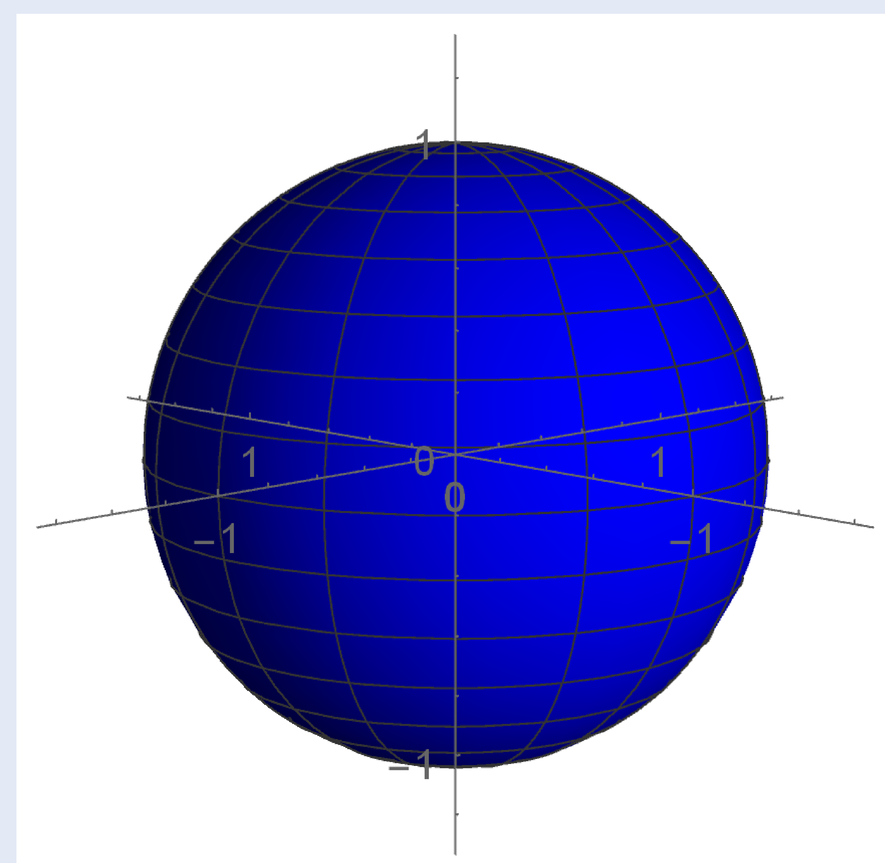

Figure: Moduli Space of $\mathbb{C}$ in $\mathbb{H}$

Along with this visual representation, we found that these planes were shuffled amongst themselves by the inner automorphisms of $\mathbb{H}$. All inner automorphisms of $\mathbb{H}$ can be represented as conjugation by an element of the 3-sphere S^3, which amazingly, is itself a (simply connected) subgroup of $\mathbb{H}$. This is summarized by the short exact sequence:

$$1 \longrightarrow \{\pm 1\} \longrightarrow S^3 \longrightarrow SO_3 \longrightarrow 1$$

Generalized Quaternions and Wedderburn

We first characterized the planes in $\mathbb{H}$, but the construction of $\mathbb{H}$ can be generalized. To study all CSAs over $\mathbb{R}$, we introduce the generalized quaternions:

Generalized Quaternions

Let $\mathbb{F}$ be a field and $a, b \in \mathbb{F}$

$$A_{a,b}(\mathbb{F}) = \left\{ t + xi + yj + zk : \begin{array}{r} t, x, y, z \in \mathbb{F}, \\ i^2 = a, j^2 = b, \\ ij = -ji = k \end{array} \right\}$$

Wedderburn's theorem tells us that every CSA is either a division algebra or a matrix ring over a division algebra. We also have a set of isomorphisms between different generalized quaternions. We can then show that there are only two classes of 4D generalized quaternions over $\mathbb{R}$, namely: $A_{-1,-1}(\mathbb{R}) \simeq \mathbb{H}$ and $A_{1,1}(\mathbb{R}) \simeq M_2(\mathbb{R})$.

Commutative Planes in the 2x2 Matrices

Any two linearly independent matrices in $M_2(\mathbb{R})$ span a plane, and just like in $\mathbb{H}$, the plane may itself be a number system, this time under matrix multiplication. As in $\mathbb{H}$, we aim to construct a moduli space for these planes. However, we found more types of planes, not just $\mathbb{C}$, but additionally the *split plane* $\mathbb{R} \times \mathbb{R}$, and a third, degenerate plane type, which we call the *nilpotent plane*.

We can construct individual moduli spaces for the three plane types based on their characteristic elements (square roots of -1, nontrivial idempotents, and nontrivial nilpotents, respectively).

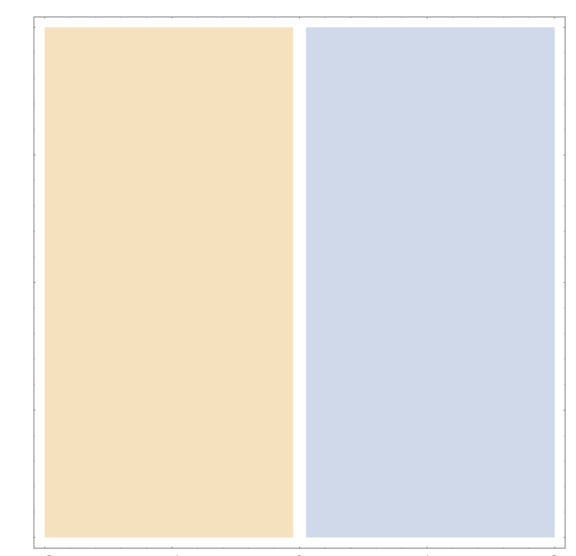

Figure: Moduli Space of $\mathbb{C}$ in $M_2(\mathbb{R})$

Pictured above is the moduli space of complex planes in $M_2(\mathbb{R})$. Each point not on the y-axis represent one such complex plane. The two halves represent two orbits under the (transitive) conjugation action of the simply connected group $SL_2(\mathbb{R})$.

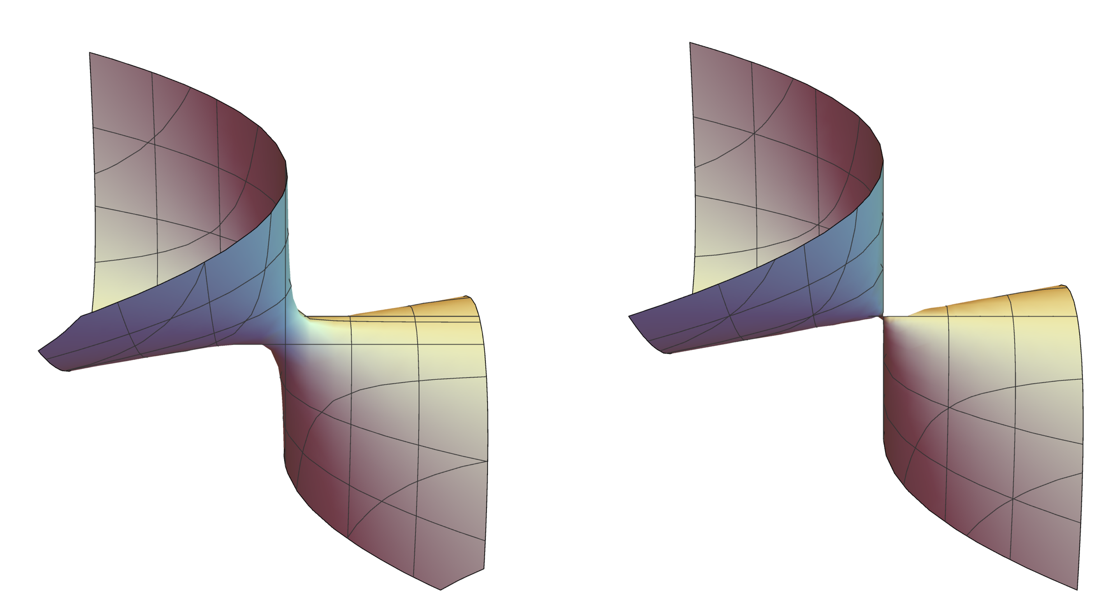

Figure: Moduli Space of $\mathbb{R} \times \mathbb{R}$ (Left) and Nilpotents in $M_2(\mathbb{R})$ (Right)

Similar to the complex planes, the nilpotent planes form two orbits under the action of $SL_2(\mathbb{R})$. In contrast, the copies of $\mathbb{R} \times \mathbb{R}$ form a single orbit.

Finally, we combine these three moduli spaces into a single moduli space representing all three types of planes. This is summarized by a short exact sequence analogous to the one that appears for the quaternions:

$$1 \longrightarrow \{\pm 1\} \longrightarrow SL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R}) \longrightarrow 1$$

Moduli Space of Planes in 2x2 Matrices

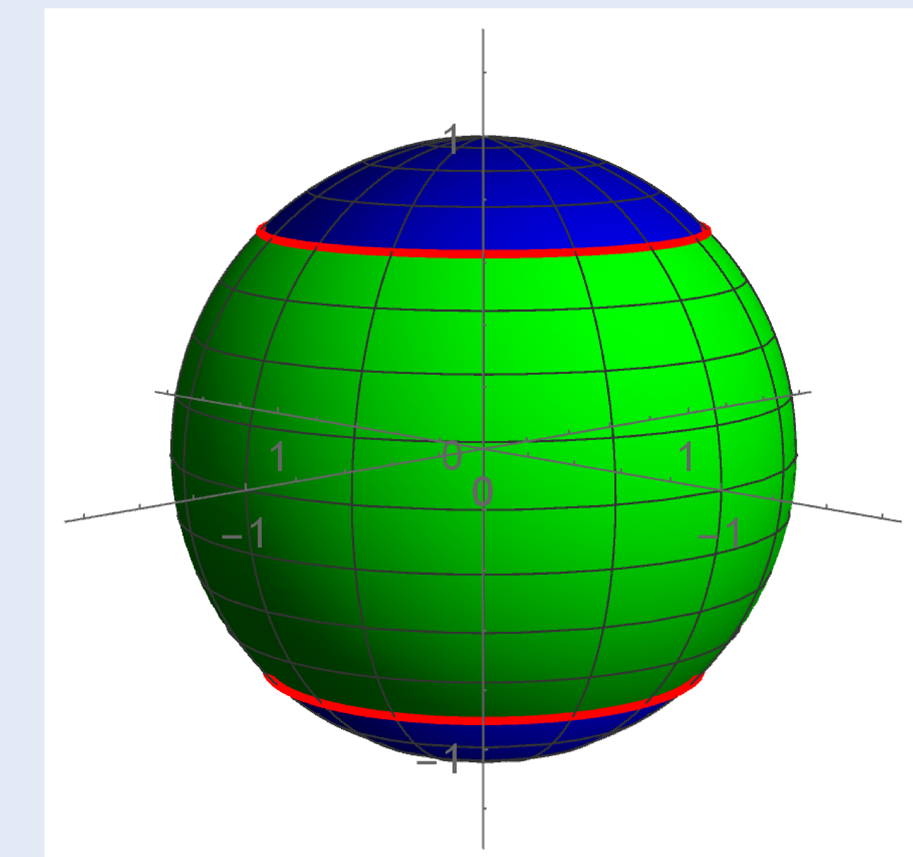

Figure: Moduli Space of Planes inside $M_2(\mathbb{R})$

In the sphere moduli space above, each color corresponds to a different type of plane. Therefore this moduli space defines a ***probability distribution*** for the different types of planes inside of $M_2(\mathbb{R})$, based on the ratio of surface areas. That is, if we were to pick a plane at random, the probability that it would be $\mathbb{C}$, $\mathbb{R} \times \mathbb{R}$, or nilpotent would be $1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$, and 0 respectively.

Conclusions

We determined that the set of all complex planes in $\mathbb{H}$ are naturally represented by the points on a sphere. We discovered 3 types of planes in $M_2(\mathbb{R})$, whose union is all of $M_2(\mathbb{R})$. As in $\mathbb{H}$, we found that each plane is naturally represented by the points on a sphere, so that again the sphere is a moduli space for the planes of $M_2(\mathbb{R})$. The three plane types take up different areas on the sphere, and determines a probability distribution for the three plane types.

Further Work

1. Give a constructive proof of Wedderburn's Theorem for the generalized quaternions over an arbitrary field.
2. Investigate the structure of commutative subalgebras of generalized quaternions over $\mathbb{Q}$, $\mathbb{Q}_p$, and finite fields $\mathbb{F}_{p^n}$.
3. Establish a unified theory for the structure of commutative subalgebras of the generalized quaternions over arbitrary fields.

Acknowledgements

Special thanks to the Bill and Linda Frost Fund for support. This research was conducted by Frost Research Fellows under the Frost Undergraduate Research Award.